

# Solution to the Hard Problem

By Philip Thomas, January 2011

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Before I begin, I would like to point out that I did not solve this on my own. I have spoken to many friends and colleagues about this, and even saw the answer posted online. Though I did not understand the answer when I first read it, it did point me in the correct direction, without which I would probably would not have come to the solution. Also, I am not entirely certain that this answer and its description are correct. I tried to write the answer up in a manner that is accessible to an undergraduate computer scientist with the help of Wikipedia. If I have bastardized mathematical notation, please forgive me.

Let  $\mathcal{S}$  be the set of all possible hattings (each of which is a string of countably infinite binary numbers representing whether the  $i$ th person has a black or white hat). Let's make a flawed assumption, then come back and try to fix the mistake. Let's assume that there is a bijection from  $\mathcal{S}$  to  $\mathbb{N}$ , the natural numbers. This is not true because  $|\mathcal{S}| = \aleph_1$ , while  $|\mathbb{N}| = \aleph_0$ . However, if such a bijection existed, we could number the hattings. Let's assume that the correct hatting number is  $h$ .

The  $i$ th individual sees the values of everyone else's hat, but does not know his or her own. Thus, he knows that the correct hatting number is either  $h_i^1$  or  $h_i^2$ , where  $h_i^1$  and  $h_i^2$  denote identical hattings except that they differ on the color of the  $i$ th person's hat. Also notice that for every person, one of these two hattings is the current one,  $h$ , and the other is unique. By unique, I mean that the  $i$ th person is the only one considering that hatting number, because it has the incorrect color for the  $i$ th person, which everyone else would have observed.

Let the  $i$ th person select the hat color that is consistent with the hatting with lesser value. That is, if  $h_i^1 < h_i^2$  then the  $i$ th person selects the hat color he is assigned in the  $h_i^1$ th hatting. Otherwise he selects the color from the  $h_i^2$ th hatting. By doing this, we know that there are only  $h - 1$  hattings with a smaller value than the correct hatting  $h$ , and because only one person could possibly guess each of these (remember that the incorrect guess that each person has is unique), at most  $h - 1$  people will guess incorrectly, as everyone else's lesser of  $h_i^1$  and  $h_i^2$  will be the correct hatting.

Ok, that's dandy, but we started with a false assumption—that there is a mapping from  $\mathcal{S}$  to  $\mathbb{N}$ . To fix this, we assume the axiom of choice<sup>1</sup>, which means that there exists a well-ordering for set  $\mathcal{S}$ . Thus, each person can select the least element from the set  $\{h_i^1, h_i^2\}$ . However, our solution remains broken because now there may be an infinite number of  $h_i^1$  and  $h_i^2$  that are less than  $h$ , so infinite people may guess incorrectly.

To fix this, we modify our strategy. First, we order the people (again with a well-ordering). We call each person's number his *rank*. Each person guesses that his hat is consistent with  $h_i$ , where  $h_i$  is now the least valued hatting (according to the well-ordering), that is consistent with the hat colors of all the people of higher rank. Let  $p_i$  denote the  $i$ th person. Notice that if  $p_i < p_j$ , then  $h_i \geq h_j$ , otherwise  $h_i$  would have been selected by  $p_j$ .

Also, if  $h_i \neq h$  (by this we mean that  $h_i$  differs from  $h$  on the color of the  $i$ th person's hat) and  $p_j < p_i$ , then  $h_j > h_i$ . Notice that  $h_j$  is strictly greater than  $h_i$ . This is because  $h_i$  is the least hatting that is correct for all  $p_k, p_k > p_i$ . If it is wrong about the  $i$ th person's hat, then all  $p_j, p_j < p_i$ , will know that it is wrong, because they see that the  $j$ th person's hat does not match. Thus, when  $p_j$  selects the minimum hatting that is consistent with all  $p_k, p_k > p_j$ , it cannot select

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<sup>1</sup>If you're not familiar with the axiom of choice or well-orderings, see the Wikipedia articles.

$h_i$ , as it is incorrect, though it is the minimum choice for  $p_i$ . Thus, the new choice  $h_j$  must satisfy  $h_j > h_i$ .

Next, consider the set of people who guess incorrectly,  $S'$ . Order this set from lowest to highest rank. As you walk through this set of people, the hatting values they guessed must be strictly decreasing (see previous paragraph). According to the Wikipedia page on well-orderings, if a set is well ordered "Every decreasing sequence of elements of the set must terminate after only finitely many steps." Thus, the set of incorrect hatting guesses (at most one per person) made by the people who guessed incorrectly, must be finite.